Notes and Solutions from the Starting Year 12 Problem Solving session

These notes should be read along with a copy of the slides from the 17th August session. The key message is: **keep trying to solve problems** and have fun — if the ones here were too difficult, find some easier ones to start with; and if they were too easy, find some harder ones. If you can't find any, your teacher can probably help, or email us at `fmspwales@swansea.ac.uk` and we'll find you some! Trying to solve problems will make you better at maths and is fun.

N.B. Don't feel any obligation to read the rest of this document — just read the bits and solutions that you're interested in. Any questions, comments or corrections are very welcome, send them to the email above. The "for the enthusiast" sections may need you to work hard with a pen and paper to understand them.

Slide 3: 1,2,3,4 number

There is a small prize for the person who gets furthest on the challenge of representing the positive integers starting at 1, using

1. the four numbers 1,2,3 and 4 once each
2. combined with as many uses of the operators $+$, $-$, $\times$ and $\div$, and pairs of brackets.

We're not allowing you to combine the numbers into a new number, such as 12, or to use power operations such as 2^3. In the event of a draw, we'll chose one entry at random. Closing date for entries 5pm Friday 27th August.

For the Python enthusiast:

I mentioned that it would be possible to write a program to find all the answers. I'll do this and send it out after the competition closes. My thoughts on how to do it are below, but I'd be interested to see anyone else's program.

A plausible approach is to generate all possible permutations of the four numbers 1,2,3,4, and all possible choices of the three operators to be used to join them, and all possible orders of applying those three operators (this corresponds to using brackets). There are useful tools for this in the Python `itertools` module. For each possible choice, evaluate the expression, perhaps using the Python `eval` function. Then store the result and expression in a dictionary.

Slide 8: Divisibility tests for the connoisseur

Why does this test work for divisibility by 19?

69768 can be written as $10 \times 6976 + 8$. If $10 \times 6976 + 8$ is divisible by 19 then so is twice that number i.e. $20 \times 6976 + 2 \times 8$. This can be written $(19 + 1) \times 6976 + 2 \times 8$ and expanded as $19 \times 6976 + 6976 + 2 \times 8$. Since the first term is a multiple of 19, we only need to worry about whether the remaining bit is divisible by 19 i.e.

$$69768 \text{ is divisible by 19 if and only if } 6976 + 2 \times 8 \text{ is.}$$

Question: is this sort of test only available for prime numbers?
Answer: there are these sort of tests available for any number not divisible by 2 or 5. However doing the test may be harder than just doing the division! I think for 7 division is easier, for 19 the test is easier.

For the enthusiast

A really useful tool for examining divisibility, and many other things e.g. understanding the RSA cipher which enabled e-commerce on the internet, is *modular arithmetic*. There are many good explanations of this on the web, or in any introductory book on a branch of maths called *number theory*. Here's a rapid explanation:

Choose a positive integer, 5 say, and group the integers into 5 columns:

...	-4	-3	-2	-1
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	...		

If we choose two numbers from the column with 1 in and add them, we'll always get something in the column with 2 in. In fact, if we choose a representative from any specific column, and a representative from any other specific column and add them, they always end up in the same column e.g. $6 + 3 = 9$, $1 + 13 = 14$, $-4 + 8 = 4$ etc. So anything in the column of 1 plus anything in the column of 3 ends up in the column of 4. We call the columns *equivalence classes* because all the members in a column are equivalent. We often choose a set of *representatives*, one per column, usually 0, 1, 2, 3 and 4. With 5 such columns we talk of arithmetic *modulo 5*, and denote it $\mathbb{Z}_5$. We write things like

$$1 + 3 \equiv 4 \pmod 5$$
$$1 + 2 \equiv 3 \pmod 5$$
$$4 + 3 \equiv 2 \pmod 5$$

Note we usually use $\equiv$ instead of $=$. Also addition "wraps round" in the last example. We can also do subtraction and multiplication, and raising to integer powers. Can we do division?

Well, yes and no. To divide 3 by 2 in this world we don't just say it's 1.5 — that doesn't have an obvious meaning here, since the only "numbers" we have are 0, 1, 2, 3 and 4. Instead of dividing by 2, we think of multiplying by "$\frac{1}{2}$". "$\frac{1}{2}$" is the number that when multiplied by 2 gives 1. We call it the *multiplicative inverse* of 2, and sometimes write it 2^{-1}. In $\mathbb{Z}_5$ $2^{-1} \equiv 3$. However ...

Working modulo a prime, all non-zero numbers have multiplicative inverses. But when modulo a composite number *we can have two numbers that multiply to give zero!* For example, in $3 \times 4 \equiv 0 \pmod{12}$. If we insist multiplication obeys "normal rules", this leads to the conclusion that 3 and 4 can't have multiplicative inverses. But 5 does have a multiplicative inverse (itself) since $5 \times 5 \equiv 1 \pmod{12}$.

Modular arithmetic has some interesting and useful results e.g. Fermat's Little Theorem says if p is prime $a^p - a \equiv 0 \pmod{p}$. So for example we can know that $2^{17} - 2$ is divisible by 17. Wilson's theorem says $(n-1) \times (n-2) \times \ldots 2 \times 1 \equiv -1 \pmod{n}$ if and only if n is prime. So for example $4 \times 3 \times 2 \times 1 \equiv 24 \equiv -1 \pmod{5}$, but $5 \times 4 \times 3 \times 2 \times 1 \equiv 120 \equiv 0 \pmod{6}$. You can use Wilson's theorem to show that any prime of form $4n + 1$ can be written as the sum of two squares. We can also use modular arithmetic to look at how our divisibility tests are working:

$$\begin{array}{rccl} & 10a + b & \equiv & 0 \pmod{19} \\ \Leftrightarrow & 10^{-1} \times 10a + 10^{-1}b & \equiv & 10^{-1} \times 0 \pmod{19} \\ \Leftrightarrow & a + 10^{-1}b & \equiv & 0 \pmod{19} \end{array}$$

And $10^{-1} \equiv 2 \pmod{19}$.

Slide 9: Hot dog, jumping frog

You spotted in the session that $n(n+2)$ seemed to be the least moves needed to swap n frogs of each colour, blue and pink on the Nrich website. But we realised that we do need to prove this result to be certain about it. We said we could break the proof into two parts: show that at least $n(n+2)$ moves are needed; and show that there was a definite way to do it in $n(n+2)$. I claimed to prove that $n(n+2)$ was a minimum but I realise my argument was flawed! (sorry) Details below.

For the enthusiast

My argument was that the total number of squares moved through/over by the n frogs was $2n(n+1)$ and we need at least n^2 jumps to move the frogs through each other, "therefore" we need $2n(n+1) - 2n^2 = 2n$ slides, so $n^2 + 2n = n(n+2)$ moves in all. BUT what about if we did more than n^2 jumps? Since each jump moves 2 squares, we'd need two less slides, and the number of moves would drop by 1. Oops!

For example if we have 2 frogs a side, but *two* spaces in the middle, so PP__BB, then my "proof" says we move 16 squares in total and need at least 4 jumps, so "need" $16-2\times4=8$ slides, so need 12 moves. But actually we can reach _PP_BB in 1 move (a jump), reach _BB_PP in 8 more moves, and reach PP__BB with a final jump, so 10 moves is possible.

One way to mend the proof would be to convince ourselves in the original situation with a single empty space that a frog cannot ever jump one of it's own colour without causing a blockage. So here we go:

Lets consider a P (a pink frog), and a puzzle with a single space, and P takes part in a solution to the puzzle. If it jumps another P, it will form a PP configuration with the space behind them. Anything in front of them will be stuck, since there's no space there, and no way to jump a PP.

Therefore before a P jumps another P, all the Bs must have got past. This in turn means that the jumping P has moved from its starting position. Assume it's the first P to jump one of its own. Then it got to its current position, as the back P of a PP, by itself moving or jumping. Which means the space is now behind it. So it can't jump the P in front of it as there's no space. So Ps don't jump Ps in a solution. Similarly Bs don't jump Bs. So we're done, and the proof is now valid (hooray!).

A different neat proof I saw on the Internet (for which I've unfortunately lost the reference, but which also needs the fact that frogs don't jump their own colour in a working solution is a follows): In the initial configuration with an equal number n of frogs P...P_B...B, there are a number of problems to solve: each P is the wrong side of each B; each P is on the wrong side of the space; and each B is on the wrong side of the space. That's n^2+2n problems. Each move we make reduces the total problems by one: a slide moves the space behind a frog; jumping over an opposite colour fixes the order of those frogs, moves the space behind the jumper BUT moves it in front of the jumpee. So to fix the n^2+2n we need n^2+2n moves. Tah-dah! The remaining question then is whether there is always a sequence of $n(n+2)$ moves without getting stuck.

Proof that $n(n+2)$ moves is possible:

The proof here is adapted from that in *Problem Solving — the Creative Side of Mathematics* by Derek Holton, where it is credited to Paul Schulter. It uses a useful technique called "Proof by Induction" that you will see if you do Further Maths A- or AS-level.

Suppose we know a way to change the initial n frog P...P_B...B pattern into an interleaved arrangement _PBPBP...PB in $\frac{n(n+1)}{2}$ moves. Then by symmetry we also know a way to get to PB...PB_ in this number of moves too (just using the mirror image version of our moves). The supposition is definitely true for $n=1$ since we use one move to change P_B to _PB, and $\frac{1(1+1)}{2}=1$.

If instead we had $n+1$ frogs then we could use our n frog version on the middle n frogs of both sides to get P_PBPB...PBB in $\frac{n(n+1)}{2}$ moves. Jumping the leftmost n Bs and sliding the last B would use $n+1$ more moves to reach PBPB...PB_ in a total of $\frac{(n+1)(n+2)}{2}$ moves.

So if the supposition is true for n, it's true for $n+1$. Since the result is true for $n=1$, hence for $n=2$, hence ... any positive integer n.

Our solution to the original problem is then to interleave, to jump all n frogs of the colour which can, then to uninterleave doing the interleave procedure moves in reverse. This is $\frac{n(n+1)}{2}$ moves plus n plus $\frac{n(n+1)}{2}$ more, so $n(n+1)+n$ i.e. $n(n+2)$ and we're done. But you may need to work through the details of some of the steps I've sketched.

Slide 11: Water pouring

We skipped this slide. Have a go at the problems before reading the solutions.

The 3, 5, 8 unit cup problem

A solution is below (there is another). The water in each cup is shown, after the previous pouring operation has finished.

0 0 8
0 5 3
3 2 3
0 2 6
2 0 6
2 5 1
3 4 1
0 4 4

There is a nice way to visualise this solution as a point moving inside an equilateral triangle. See *Puzzles and Paradoxes* by T.H. O'Beirne for details, or `en.wikipedia.org/wiki/Water_pouring_puzzle`.

The 5, 11, 13, 24 unit cup problem

A solution is below. One technique that's sometimes helpful is to work *backwards*, aiming to get containers completely full or completely empty.

0 0 0 24
0 0 13 11
5 0 8 11
0 0 8 16
5 0 3 16
0 0 3 21
0 3 0 21
0 3 13 8

5 3 8 8
0 8 8 8

Slide 12: The Census-Taker problem

I like this problem as it seems crazy from the conversation that the census-taker can work out the details. What can be the relevance of the oldest daughter being called Annie and liking dogs?! However if you work out the implications of each of the mother's statements, it makes sense!

The mother says the product of her daughters' ages is 36, so their possible ages are:

1 1 36
1 2 18
1 3 12
1 4 9
1 6 6
2 2 9
2 3 6
3 3 4

It's important to work systematically and find all the possibilities. The mother's next statement that she could reveal the sum of the ages, but that wouldn't be enough information is actually a big clue. The possible sums are: 38, 21, 16, 14, 13, 13, 11, 10, so this means the ages are either 1 6 6 or 2 2 9.

So now we can see the third clue isn't about the daughter's name or her liking dogs, but the fact that she is *the* oldest. So we can't have the solution with two daughters aged 6 since there is not a unique oldest child here. (If this was the real world rather than a maths problem, we could quibble about one of the two 6 year olds having to be born before the other! But that fails to appreciate the elegance of the puzzle.)

Slide 13: The Logic Grid puzzle

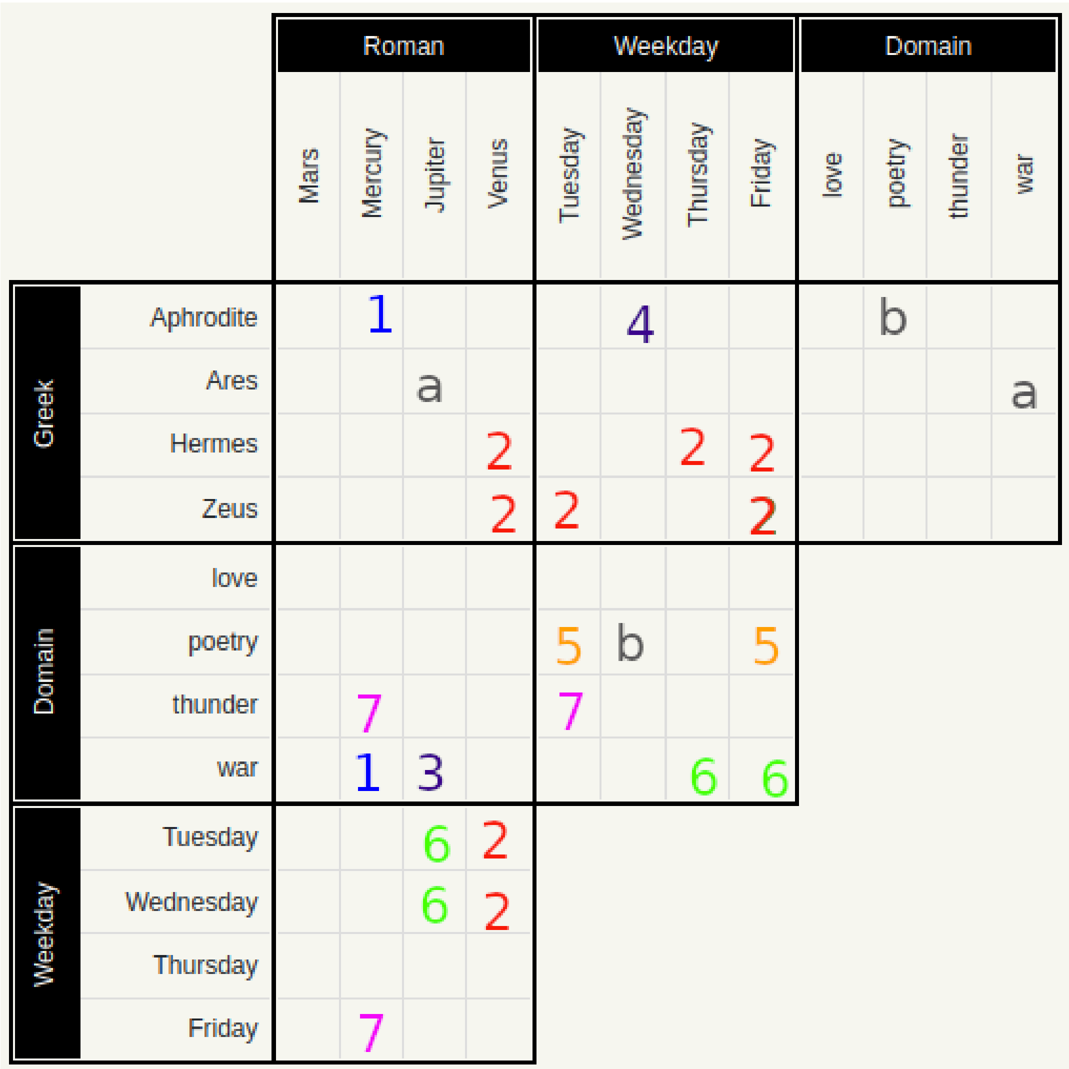
Logic problems aren't really problems where you don't know how to approach them, after the first few times. However they are fun and they are good for practicing logical thinking. We got started on the Ancient Gods logic puzzle at `www.brainzilla.com/logic/logic-grid/ancient-gods/`. Below is my go at solving it: the numbers in circles in the grid represent crosses, deduced from the clue of that number. And one of the two squares marked "a" must be a tick, from clue 3, and one of the two squares marked "b" must be a tick, from clue 4.

We have three rules that aren't fully marked on the grid:

- 2: Zeus is the day after Hermes and day before Venus
- 6: Jupiter is two days after War
- 7: Mercury is one day before Thunder

I unfortunately can't see how to get beyond this point without guessing and checking. If you can see it, let me know! The best guess to make is one that has fewest options and that forces the most consequences e.g. Zeus' day can only be Wednesday or Thursday and forces Hermes and Venus. Let's guess the wrong option and see what happens:

- Guess Zeus is Wednesday, so from rule 2 Hermes is Tuesday and Venus is Thursday
- Then Jupiter is Friday, and from rule 6 War is Wednesday, and rule 4 means Aphrodite is Poetry
- Poetry is on Thursday and hence Thunder on Friday and Love on Tuesday
- Rule 7 then says Mercury is Thursday IMPOSSIBLE since we've crossed this out.



If instead we guess the other option:

- Guess Zeus is Thursday, so from rule 2 Hermes is Wednesday and Venus is Friday
- Then Jupiter is Thursday so Jupiter must be Zeus
- Rule 3 says Ares is War and rule 6 says War is Tuesday
- War can only be Mars or Venus, but Venus is Friday, so War is Mars and Mars is Tuesday too, making Mercury Wednesday.
- Rule 7 means Thunder is Thursday hence Thunder is Jupiter and Zeus
- Poetry is Wednesday, hence by rule 4 Aphrodite is not Poetry, and must be Love, leaving Hermes as Poetry, and Love on Friday

We now have the mapping from Romans to Days and Days to Domains and Domains to Greeks, so we can use this to fill in the rest.

Slide 14: Socks in the Drawer

Again we didn't do this one — have a go before reading the solution below.

This is a case where algebra can help us. Let r be the number of red socks and b the number of black ones. The question tells us that

$$\frac{r}{r+b} \times \frac{r-1}{r+b-1} = \frac{1}{2} \tag{1}$$

or

$$2r(r-1) = (r+b)(r+b-1)$$

so one way to tackle this is to look through the numbers of of form $n(n-1)$ for one that is double another. For $n = 2, 3, 4, \ldots$ we have $2, 6, 12, \ldots$ so $r = 3$, $b = 1$ works. Searching further along the list will locate $r = 15$ and $b = 6$.

For the enthusiast

Can we reduce our searching at all? The book I got the puzzle from (see slides) points out that to get (1) to work we need

$$\frac{r}{r+b} > \frac{1}{\sqrt{2}}$$

and

$$\frac{r-1}{r+b-1} < \frac{1}{\sqrt{2}}$$

and solving these gives

$$(\sqrt{2}+1)b < r < (\sqrt{2}+1)b$$

and trying the appropriate r value for $b = 2, 4, 6, \ldots$ finds the second value in 3 steps rather than 21. The book also notes the solutions are related to something called *Pell's Equation*.

Slide 15: The Monty Hall problem

We didn't do this one, but it is a classic problem. So have a go, before reading the solution below.

The "obvious", but wrong, argument is that since there are two doors remaining, the car could be behind either, so swapping wouldn't help. BUT the probability of the car being behind each door surprisingly isn't the same. To see this:

There's a $\frac{1}{3}$ chance of car being behind the door you originally pick. Opening the door doesn't change that. The car had a $\frac{2}{3}$ chance of being behind a non-picked door. Opening the door has told us, if the car is behind a non-picked door, it is definitely behind the closed one. So switching your choice doubles your chances.

You may not find the above convincing, but it is true. Perhaps you may want to investigate other ways to convince yourself that swapping is the better decision.